On the IR behaviour of the Landau-gauge ghost propagator

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ABSTRACT: We examine analytically the ghost propagator Dyson-Schwinger Equation (DSE) in the deep IR regime and prove that a finite ghost dressing function at vanishing momentum is an alternative solution (solution II) to the usually assumed divergent one (solution I). We furthermore find that the Slavnov-Taylor identities discriminate between these two classes of solutions and strongly support the solution II. The latter turns out to be also preferred by lattice simulations within numerical uncertainties.

KEYWORDS: Confinement, Lattice Gauge Field Theories, Nonperturbative Effects, Lattice Quantum Field Theory.

Contents

1. Introduction

How to deal with the IR behaviour of QCD ? There are three main types of approach:

- Dyson Schwinger equations (DSE) and especially the untruncated one concerning the ghost propagator.

- Ward-Slavnov-Taylor identities (WSTI)

- Lattice QCD simulations (LQCD).

   Until a few years ago, there was a clear contradiction between the standard DSE solution and LQCD results. If we call $F(q^2)$ $(G(q^2))$ the ghost (gluon) dressing function, the standard DSE solution (later labelled as solution I) predicts that $F^2(q^2)G(q^2)$ goes to a non-vanishing constant when $q^2 \to 0$ (see for instance [1] and references therein). LQCD indicates on the contrary in an unambiguous way that $F^2(q^2)G(q^2) \to 0$ when $q^2 \to 0$ [2, 3]. The standard solution implies [4] also that $G(q^2)/q^2$ does not diverge when $q^2 \to 0$ while $F(q^2)$ diverges at least as fast as $(q^2)^{-1/2}$. Regarding lattice QCD results, they have long been compatible with an IR-diverging $F(q^2)$, although definitely at a much slower pace. This discrepancy has been tentatively charged to different types of lattice

artifacts. However more recent LQCD data obtained in large volume simulations [5, 6] show that under those conditions the ghost dressing function IR exponent α_F (assuming $F(q^2) \underset{q^2 \to 0}{\simeq} (q^2)^{\alpha_F}$) lies in the vicinity of 0.

Now, it was proven in [3, 7, 8] that:

- there exists a second class of solutions to the DSE (later labelled as solution II) which implies that $F(q^2)$ goes to a non-vanishing constant when $q^2 \to 0$ and does not constrain $F^2(q^2)G(q^2)$.

- the WSTI implies under very plausible assumptions that $F(q^2)$ goes to a non-vanishing constant when $q^2 \to 0$, which imposes the solution II of DSE.

Thus, **the convergence of the three methods towards a finite non-vanishing ghost dressing function is very impressive.**
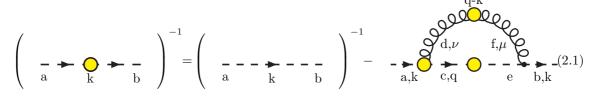
Furthermore, a recent numerical study of the DSE using the LQCD gluon input finds that both cases of solutions (I and II) are found depending on the strong coupling constant which is a free parameter in this exercise [9]. Solutions exist when the coupling constant is smaller than (or equal to) a critical value. In the general case the solutions which come out belong to type II, but for the critical coupling constant one finds the solution I. It was also proved that for an appropriate coupling constant the resulting ghost dressing function (belonging to class II) fits very well with lattice results.

Concerning the gluon propagator the analytic methods are not so constraining. WSTI, under a regularity hypothesis for the longitudinal-longitudinal-transverse gluon vertex function, predicts a divergent gluon propagator when $q^2 \to 0$ [3, 10] while LQCD seems to point towards a finite non-vanishing gluon propagator at $q^2 = 0$ (see, for instance, [11]). A very slow divergence of the gluon propagator, not easy to see in LQCD, might solve this discrepancy.

In this paper we wish to present an analytic study of the ghost propagators of both solutions I and II in the deep infrared in the context of the DSE. We also will carefully scrutinize the relationship between DSE, WSTI and LQCD solutions. In section 2, the ghost propagator DSE is properly renormalised and analysed in the deep IR regime. The two types of solutions are obtained in section 3 and their implications put clearly on the table. In section 4, we discuss what WSTI tells us and section 5 is devoted to briefly review the LQCD results for the ghost propagator. We conclude in section 6. In appendix A we show how the ghost propagator DSE that we exploit in the next section can be generally inferred from WSTI.

2. The ghost propagator Dyson-Schwinger equation

We will examine the Dyson-Schwinger equation for the ghost propagator (GPDSE) which can be written diagrammatically as

$$(2.1)$$

i.e., denoting by $F^{(2)}$ (resp. $G^{(2)}$) the full ghost (resp. gluon) propagator,

$$(F^{(2)})^{-1}_{ab}(k) = -\delta_{ab}k^2 - \tag{2.2}$$
$$-g_0^2 f_{acd}f_{ebf} \int \frac{d^4q}{(2\pi^4)} F^{(2)}_{ce}(q)(iq_{\nu'})\widetilde{\Gamma}_{\nu'\nu}(-q,k;q-k)(ik_\mu)(G^{(2)})^{fd}_{\mu\nu}(q-k),$$

where $\widetilde{\Gamma}$ stands for the bare ghost-gluon vertex,

$$\widetilde{\Gamma}^{abc}_\nu(-q,k;q-k) = ig_0 f^{abc} q_{\nu'}\widetilde{\Gamma}_{\nu'\nu}(-q,k;q-k)$$
$$= ig_0 f^{abc}\left(q_\nu H_1(q,k) + (q-k)_\nu H_2(q,k)\right), \tag{2.3}$$

where q and k are respectively the outgoing and incoming ghost momenta and g_0 is the bare coupling constant. Let us now consider eq. (2.2) at small momenta k. After applying the decomposition for the ghost-gluon vertex in eq. (2.3), omitting colour indices and dividing both sides by k^2, it reads

$$\frac{1}{F(k^2)} = 1 + g_0^2 N_c \int \frac{d^4q}{(2\pi)^4}\left(\frac{F(q^2)G((q-k)^2)}{q^2(q-k)^4}\left[\frac{(k\cdot q)^2}{k^2} - q^2\right]H_1(q,k)\right). \tag{2.4}$$

It should be noticed that, because of the transversality condition, H_2 defined in eq. (2.3) does not contribute for the GPDSE in the Landau gauge.

2.1 Renormalization of the Dyson-Schwinger equation

The integral equation eq. (2.4) is written in terms of bare Green functions. It is actually meaningless unless one specifies some appropriate UV-cutoff,[1] Λ, and performs the replacements $F(k^2) \to F(k^2,\Lambda)$ .... It can be cast into a renormalized form by dealing properly with UV divergencies, i.e.

$$g_R^2(\mu^2) = Z_g^{-2}(\mu^2,\Lambda)g_0^2(\Lambda)$$
$$G_R(k^2,\mu^2) = Z_3^{-1}(\mu^2,\Lambda)G(k^2,\Lambda)$$
$$F_R(k^2,\mu^2) = \widetilde{Z}_3^{-1}(\mu^2,\Lambda)F(k^2,\Lambda), \tag{2.5}$$

where μ^2 is the renormalization momentum and Z_g, Z_3 and $\widetilde{Z}_3$ the renormalization constants for the coupling constant, the gluon and the ghost respectively. Z_g is related to the ghost-gluon vertex renormalization constant (defined by $\widetilde{\Gamma}_R = \widetilde{Z}_1\Gamma_B$) through $Z_g = \widetilde{Z}_1(Z_3^{1/2}\widetilde{Z}_3)^{-1}$. Then Taylor's well-known non-renormalization theorem, which states that $H_1(q,0) + H_2(q,0) = 1$ in Landau gauge and to any perturbative order, can be invoked to conclude that $\widetilde{Z}_1$ is finite. We recall that the renormalization point is arbitrary, except for the special value $\mu = 0$ which cannot be chosen without a loss of generality (see, in this respect, the discussion in ref [12]). Thus,

$$\frac{1}{F_R(k^2,\mu^2)} = \widetilde{Z}_3(\mu^2,\Lambda) + N_C\widetilde{Z}_1 g_R^2(\mu^2)\Sigma_R(k^2,\mu^2;\Lambda) \tag{2.6}$$

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[1]We have written for simplicity the UV cutoff as a hard cut-off. It is preferable to use a gauge invariant regularization procedure in view of the advantage of exploiting Ward-Slavnov-Taylor identities (see section 4). In practice we will derive our results from the subtracted GPDSE which incorporates gauge invariant UV regularisation.

where

$$\Sigma_R(k^2, \mu^2; \Lambda) = \int^{q^2 < \Lambda^2} \frac{d^4 q}{(2\pi)^4} \qquad (2.7)$$

$$\times \left(\frac{F_R(q^2, \mu^2) G_R((q-k)^2, \mu^2)}{q^2 (q-k)^4} \left[\frac{(k \cdot q)^2}{k^2} - q^2 \right] H_{1,R}(q, k; \mu^2) \right).$$

One should notice that the UV cut-off, Λ, is still required as an upper integration bound in eq. (2.7) since the integral is UV-divergent, behaving as $\int dq^2/q^2 (1 + 11\alpha_S/(2\pi) \log(q/\mu)))^{-35/44}$. In fact, the cut-off dependence this induces in Σ_R cancels[2] against the one of $\widetilde{Z}_3$ in the r.h.s. of eq. (2.6), in accordance with the fact that the l.h.s. does not depend on Λ.

Now, we will apply a MOM renormalization prescription. This means that all the Green functions take their tree-level value at the renormalization point and thus:

$$F_R(\mu^2, \mu^2) = G_R(\mu^2, \mu^2) = 1. \qquad (2.8)$$

In the following, $H_1(q, k)$ will be approximated by a constant[3] with respect to both momenta and, provided that $H_1(q, 0) = 1$ at tree-level, our MOM prescription implies that $H_{1,R}(k, q; \mu^2) = 1$ and $\widetilde{Z}_1$ is a constant in terms of μ.

2.2 A subtracted Dyson-Schwinger equation

The renormalized GPDSE, eq. (2.6), should be carefully analysed. We aim to study the infrared behaviour of its solutions and therefore focus our analysis on the momentum region, $k \ll \Lambda_{\mathrm{QCD}}$, where the IR behaviour of the dressing functions (presumably in powers of the momentum) is supposed to hold. One cannot forget, though, that the UV cut-off dependences in both sides of eq. (2.6) match only in virtue of the previously mentionned relation between the ghost and gluon propagator anomalous dimension and the beta function.

However, in order not to have to deal with the UV cut-off, we prefer to approach the study of the GPDSE in the following manner: we consider eq. (2.6) for two different scales, λk and $\lambda \kappa k$ (with $\kappa < 1$ some fixed number and λ an extra parameter that we shall ultimately let go to 0) and subtract them

$$\frac{1}{F_R(\lambda^2 k^2, \mu^2)} - \frac{1}{F_R(\lambda^2 \kappa^2 k^2, \mu^2)} = N_C g_R^2(\mu^2) \widetilde{Z}_1 \left(\Sigma_R(\lambda^2 k^2, \mu^2; \infty) - \Sigma_R(\lambda^2 \kappa^2 k^2, \mu^2; \infty) \right). \qquad (2.9)$$

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[2]One can easily check that $\widetilde{Z}_3^{-1}(\mu^2, \Lambda) \Sigma_R(k^2, \mu^2; \Lambda)$ approaches some finite limit as $\Lambda \to \infty$ since the ghost and gluon propagator anomalous dimensions and the beta function verify the relation $2\widetilde{\gamma} + \gamma + \beta = 0$.

[3]This approximation is very usually used to solve GPDSE. Notice that few lattice data are available for the ghost-gluon vertex. However, in a recent computation [2] of that vertex for a zero gluon momentum, $H_1(q, q)$ appears to be approximatively constant with respect to q. Of course, more data for different kinematical configurations should be welcome to check that approximation.

Then the integral in the r.h.s. is UV-safe, thanks to the subtraction, and the limit $\Lambda \to \infty$ can be explicitly taken,

$$\Sigma_R(\lambda^2 k^2, \mu^2; \infty) - \Sigma_R(\lambda^2 \kappa^2 k^2, \mu^2; \infty) = \int \frac{d^4 q}{(2\pi)^4} \left(\frac{F(q^2, \mu^2)}{q^2} \left(\frac{(k \cdot q)^2}{k^2} - q^2 \right) \right.$$
$$\left. \times \left[\frac{G((q - \lambda k)^2, \mu^2)}{(q - \lambda k)^4} - (\lambda \to \lambda\kappa) \right] \right). \quad (2.10)$$

An accurate analysis of eq. (2.9) requires, in addition, to cut the integration domain of eq. (2.10) into two pieces by introducing some new scale q_0^2 (q_0, typically of the order of Λ_{QCD}, is a momentum scale below which the deep IR power behaviour is a good approximation),

$$\Sigma_R(\lambda^2 k^2, \mu^2; \infty) - \Sigma_R(\lambda^2 \kappa^2 k^2, \mu^2; \infty) = I_{\text{IR}}(\lambda) + I_{\text{UV}}(\lambda) \quad (2.11)$$

where I_{IR} represents the integral in eq. (2.10) over $q^2 < q_0^2$ and I_{UV} over $q^2 > q_0^2$. Only the dependence on λ is written explicitly because we shall let it go to zero with k, κ and μ^2 kept fixed. The relevance of the q_0^2 scale stems from the drastic difference between the IR and UV behaviours of the integrand. In particular, for $(\lambda k)^2 \ll q_0^2$, the following infrared power laws,

$$F_{\text{IR}}(q^2, \mu^2) = A(\mu^2) \left(q^2 \right)^{\alpha_F}$$
$$G_{\text{IR}}((q - \lambda k)^2, \mu^2) = B(\mu^2) \left((q - \lambda k)^2 \right)^{\alpha_G}, \quad (2.12)$$

will be applied for both dressing functions in I_{IR}.

Now, I_{IR} is infrared convergent if:

$$\alpha_F > -2 \qquad \text{IR convergence at } q^2 = 0$$
$$\alpha_G > 0 \qquad \text{IR convergence at } (q - k)^2 = 0 \text{ and } (q - \kappa k)^2 = 0 \quad (2.13)$$

We shall suppose in the following that these conditions are verified. Let us first consider I_{UV}. Its dependence on λ, which is explicit in the factor inside the square bracket of eq. (2.10), should clearly be even in λ: any odd power of λ would imply an odd power of $q \cdot k$ whose angular integral is zero. Since the integrand is identically zero at $\lambda = 0$ and the integral is ultraviolet convergent, it is proportional to λ^2 (unless some accidental cancellation forces it to behave as an even higher power of λ). On the other hand, after performing the change of variable $q \to \lambda q$, the IR contribution of the integral in eq. (2.10)'s r.h.s can be rewritten as:

$$I_{\text{IR}}(\lambda) \simeq \left(\lambda^2 \right)^{(\alpha_F + \alpha_G)} A(\mu^2) B(\mu^2) \int^{q^2 < \frac{q_0^2}{\lambda^2}} \frac{d^4 q}{(2\pi)^4} \left(q^2 \right)^{\alpha_F - 1} \left(\frac{(k \cdot q)^2}{k^2} - q^2 \right)$$
$$\times \left[((q - k)^2)^{\alpha_G - 2} - ((q - \kappa k)^2)^{\alpha_G - 2} \right], \quad (2.14)$$

that, as it shall be seen in the next subsection, asymptotically behaves as

$$I_{\text{IR}}(\lambda) \sim \begin{cases} \lambda^{2(\alpha_G + \alpha_F)} & \text{if} \quad \alpha_G + \alpha_F < 1 \\ \lambda^2 \ln \lambda & \text{if} \quad \alpha_G + \alpha_F = 1 \\ \lambda^2 & \text{if} \quad \alpha_G + \alpha_F > 1. \end{cases} \quad (2.15)$$

Thus, in all the cases, the leading behaviour of $I_{\text{IR}} + I_{\text{UV}}$, as λ vanishes, is given by I_{IR} in eq. (2.15). The subtracted renormalised GPDSE reads for $\alpha_G + \alpha_F \leq 1$ as:

$$\frac{1}{F_R(\lambda^2 k^2, \mu^2)} - \frac{1}{F_R(\lambda^2 \kappa^2 k^2, \mu^2)} \simeq N_C g_R^2(\mu^2) \widetilde{Z}_1 I_{\text{IR}}(\lambda), \qquad (2.16)$$

for small λ. We have assumed that H_1 is constant when varying all the momenta but (2.15), (2.16) remain true if one only assumes that H_1 behaves "regularly" for $q^2, k^2 \leq q_0^2$ (i.e. is free of singularities or, at least, of any singularity worse than logarithmic).

2.3 The ghost-loop integral

The present section is devoted to the quantitative analysis of the integral $I_{\text{IR}}(\lambda)$, defined in eq. (2.14), which gives the contribution of the ghost loop to the renormalised GPDSE eq. (2.16). If $\alpha_F + \alpha_G < 1$ it is possible to perform analytically the integral and to find a compact expression for it. In this case, one can write

$$I_{\text{IR}}(\lambda) \simeq A(\mu^2) B(\mu^2) \left(\lambda^2\right)^{(\alpha_F + \alpha_G)} \left(\Phi(k; \alpha_F, \alpha_G) - \Phi(\kappa k; \alpha_F, \alpha_G)\right) \qquad (2.17)$$

where $A(\mu^2)$ and $B(\mu^2)$ were defined in eq. (2.12) and

$$\Phi(k; \alpha_F, \alpha_G) = \int \frac{d^4 q}{(2\pi)^4} (q^2)^{\alpha_F - 1} \left((q-k)^2\right)^{\alpha_G - 2} \left(\frac{(k \cdot q)^2}{k^2} - q^2\right), \qquad (2.18)$$

provided that $\Phi(k; \alpha_F, \alpha_G)$ is not singular to let the subtraction inside the bracket and the integral operator in eq. (2.17) commute with each other. Then, following [13], we define

$$f(a, b) = \frac{16\pi^2}{(k^2)^{2+a+b}} \int \frac{d^4 q}{(2\pi)^4} (q^2)^a \left((q-k)^2\right)^b$$
$$= \frac{\Gamma(2+a)\Gamma(2+b)\Gamma(-a-b-2)}{\Gamma(-a)\Gamma(-b)\Gamma(4+a+b)}, \qquad (2.19)$$

and obtain

$$\Phi(k; \alpha_F, \alpha_G) = \frac{(k^2)^{\alpha_F + \alpha_G}}{16\pi^2} \phi(\alpha_F, \alpha_G) \qquad (2.20)$$

where

$$\phi(\alpha_F, \alpha_G) = -\frac{1}{2} \left(f(\alpha_F, \alpha_G - 2) + f(\alpha_F, \alpha_G - 1) + f(\alpha_F - 1, \alpha_G - 1)\right)$$
$$+ \frac{1}{4} \left(f(\alpha_F - 1, \alpha_G - 2) + f(\alpha_F - 1, \alpha_G) + f(\alpha_F + 1, \alpha_G - 2)\right). \quad (2.21)$$

Thus, if $\alpha_F + \alpha_G < 1$,

$$I_{\text{IR}}(\lambda) \simeq \frac{A(\mu^2) B(\mu^2)}{16\pi^2} (\lambda^2 k^2)^{\alpha_F + \alpha_G} (1 - \kappa^{2(\alpha_F + \alpha_G)}) \phi(\alpha_F, \alpha_G). \qquad (2.22)$$

We will now compute the leading asymptotic behavior of $I_{\rm IR}$ as $\lambda \to 0$ when $\alpha_F + \alpha_G = 1$. In this case, after performing in eq. (2.14) the following expansion,

$$
\left[(k-q)^2\right]^{\alpha_G-2} - \left[(\kappa k - q)^2\right]^{\alpha_G-2} \simeq (q^2)^{\alpha_G-2}(\alpha_G-2)(1-\kappa) \tag{2.23}
$$
$$
\times \left[-2\frac{q\cdot k}{q^2} + (1+\kappa)\left(\frac{k^2}{q^2} + 2(\alpha_G-3)\frac{(q\cdot k)^2}{q^4}\right)\right];
$$

and neglecting the term odd in $q_\mu \to -q_\mu$; one finds for the leading contribution

$$
I_{\rm IR}(\lambda) \simeq -k^2(1-\kappa^2)\frac{2A(\mu^2)B(\mu^2)}{(2\pi)^3}\lambda^2 \int^{q_0/\lambda} dq\, q^{2(\alpha_F+\alpha_G)-3}
$$
$$
\times \int_0^\pi d\theta \sin^4\theta \left(\alpha_G - 2 + 2(\alpha_G-3)(\alpha_G-2)\cos^2\theta\right)
$$
$$
\simeq k^2(1-\kappa^2)\frac{A(\mu^2)B(\mu^2)}{32\pi^2}\alpha_G(\alpha_G-2)\lambda^2 \ln\lambda. \tag{2.24}
$$

We do not specify the lower bound of the integral over q in eq. (2.24) because it necessarily contributes as a subleading term, once the ghost-loop integral is required to be IR safe. Then, as we advanced in eq. (2.15), $I_{\rm IR}/\lambda^2$ diverges logarithmically as λ goes to zero if $\alpha_F + \alpha_G = 1$. In fact, since eq. (2.22) is a reliable result for any $\alpha_F + \alpha_G < 1$ however close it may be to 1, such a divergence appears as a pole of a Gamma function of $\phi(\alpha_F, \alpha_G)$ in eq. (2.20).

Finally, if $\alpha_F + \alpha_G > 1$, the leading contribution for $I_{\rm IR}(\lambda)$ as λ vanishes can be computed after performing back the change of integration variable, $q \to q/\lambda$, in eq. (2.14). The first even term in eq. (2.23) dominates again the expansion after integration, but now it does not diverge. Then, if we procceed as we did in eq. (2.24), we obtain

$$
I_{\rm IR}(\lambda) \simeq -\frac{\alpha_G(\alpha_G-2)}{\alpha_F+\alpha_G-1}\frac{(q_0^2)^{\alpha_F+\alpha_G-1}}{64\pi^2}A(\mu^2)B(\mu^2)k^2\lambda^2(1-\kappa^2), \tag{2.25}
$$

for small λ and $\alpha_G + \alpha_F > 1$. It should be noticed that $I_{\rm IR}$ in eq. (2.25) depends on the additional scale q_0 introduced in eq. (2.11) to separate IR and UV integration domains. In fact, if one takes $q_0 \to \infty$, $I_{\rm IR}$ diverges. This means that, when $\alpha_F + \alpha_G > 1$, the behaviour of the IR power laws hampers their use for all momenta in the integral. The finiteness of the ghost-loop integral of the subtracted GPDSE can only be recovered after taking into account the UV logarithmic behaviour for large-momenta dressing functions.[4] Furthermore, $I_{\rm UV}$, also behaving as λ^2, should be also added in r.h.s. of eq. (2.16) in order to write the renormalised GPDSE. Thus, the dependence on λ but not the factor in front of it can be inferred from the GPDSE with only the information of the asymptotics for small-momentum dressing functions.

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[4]The scale q_0 being of the order of $\Lambda_{\rm QCD}$, power laws for $\alpha_F + \alpha_G > 1$ cannot be then solutions of the GPDSE in the MR truncation scheme corresponding to $\Lambda_{\rm QCD} \to \infty$ (see, for instance, [13]). The same argument holds also for $\alpha_F + \alpha_G = 1$, because the ghost-loop integral in eq. (2.24) diverges as $\lambda \to 0$ for any q_0 fixed as well as for $q_0 \to \infty$ for any fixed λ.

3. The infrared analysis of GPDSE solutions

The starting point for the infrared analysis will be the eq. (2.16) for small λ, where we will try to make the dependences on k, κ and λ of the two sides match each other.

3.1 The case $\alpha_F \neq 0$ (solution I)

We will first study the case $\alpha_F \neq 0$. Then, the l.h.s. of eq. (2.16) can be expanded for small λ as

$$\frac{1}{F_R(\lambda^2 k^2, \mu^2)} - \frac{1}{F_R(\lambda^2 \kappa^2 k^2, \mu^2)} \simeq \left(1 - \kappa^{-2\alpha_F}\right) \frac{\left(\lambda^2 k^2\right)^{-\alpha_F}}{A(\mu^2)} \tag{3.1}$$

and we will obtain from eq. (2.16):

$$N_C g_R^2(\mu^2) \widetilde{Z}_1 A(\mu^2) \frac{I_{\mathrm{IR}}(\lambda)}{\left(1 - \kappa^{-2\alpha_F}\right)\left(\lambda^2 k^2\right)^{-\alpha_F}} \simeq 1, \tag{3.2}$$

where the dependences on k, κ and λ of the numerator and the denominator should cancel against each other. Using for I_{IR} the form given in eq. (2.15), we find three possible situations:

- If $\alpha_G + \alpha_F > 1$, applying eq. (2.25) in eq. (3.2), we are led to the conclusion that only $\alpha_F = -1$ (and $\alpha_G > 2$) satisfies this last equation and could be an IR solution for GPDSE. However, such a solution appears to be in a clearcut contradiction with the current lattice simulations.

- If $\alpha_G + \alpha_F = 1$, there is no possible solution because the logarithmic behaviour of I_{IR} in eq. (2.24) cannot be compensated by the powerlike one in the denominator of eq. (3.2).

- If $\alpha_G + \alpha_F < 1$, eq. (2.22) combined with eq. (3.2) implies the familiar relation $2\alpha_F + \alpha_G = 0$ and we have then:

$$N_C g_R^2(\mu^2) \widetilde{Z}_1 \frac{(A(\mu^2))^2 B(\mu^2)}{16\pi^2} \phi\left(-\frac{\alpha_G}{2}, \alpha_G\right) \simeq 1, \tag{3.3}$$

An immediate consequence of this last condition is the freezing of the running coupling constant at small momentum. If the renormalization point, μ, is arbitrarily chosen to be very small in order that the dressing functions observe the power laws at $k^2 = \mu^2$, one obtains $A(\mu^2) = \mu^{-2\alpha_F}$ and $B(\mu^2) = \mu^{-2\alpha_G}$. Eq. (3.3) then reads

$$N_C g_R^2(\mu^2) \widetilde{Z}_1 \phi\left(-\frac{\alpha_G}{2}, \alpha_G\right) \simeq 16\pi^2, \tag{3.4}$$

and should be satisfied for any small value of μ. Consequently, it should remain exact as $\mu \to 0$ and provides the small-momentum limit of the running coupling (which is independent of the infrared constants for ghost and gluon dressing functions).

In particular, if $\alpha_G = 1$, one has $\phi(-1/2, 1) = 8/5$ and thus

$$N_C g_R^2(\mu^2) \widetilde{Z}_1 \simeq 10\pi^2, \tag{3.5}$$

3.2 The case $\alpha_F = 0$ (solution II)

The case $\alpha_F = 0$ is particular in that the leading contributions to the two occurrences of F in the l.h.s of eq. (2.16) cancel against each other. We have then to go one step further, taking into account the subleading terms. Defining $\widetilde{F}_{\mathrm{IR}}$ by means of $F_{\mathrm{IR}}(q^2, \mu^2) = A(\mu^2) + \widetilde{F}_{\mathrm{IR}}(q^2, \mu^2)$ we rewrite the l.h.s of eq. (2.9) as $-(\widetilde{F}_{\mathrm{IR}}(\lambda^2 k^2, \mu^2) - \widetilde{F}_{\mathrm{IR}}(\lambda^2 \kappa^2 k^2, \mu^2))/A^2(\mu^2)$ and use the known IR behaviour of $I_{\mathrm{IR}}(\lambda)$ from eq. (2.15)) in the r.h.s of eq. (2.16) to get

$$F_{\mathrm{IR}}(q^2, \mu^2) = \begin{cases} A(\mu^2) + A_2(\mu^2) q^2 \ln q^2 & \text{if} \quad \alpha_G = 1 \\ A(\mu^2) + A_2(\mu^2)(q^2)^{\alpha_F^{(2)}} & \text{otherwise.} \end{cases} \tag{3.6}$$

Furthermore, not only the subleading functional behaviour of the dressing function can be constrained but also the coefficient A_2 in eq. (3.6). In fact, if we plug this equation into the l.h.s of eq. (2.16) and expand we obtain:

$$-\frac{(A(\mu^2))^2}{A_2(\mu^2)} N_C g_R^2(\mu^2) \widetilde{Z}_1 \ I_{\mathrm{IR}}(\lambda) \simeq \begin{cases} k^2(1-\kappa^2)\lambda^2 \ln \lambda^2 & \text{if } \alpha_G = 1 \\ (\lambda^2 k^2)^{\alpha_F^{(2)}} (1 - \kappa^{2\alpha_F^{(2)}}) & \text{otherwise,} \end{cases} \tag{3.7}$$

Let us consider now in more detail the three possible cases.

- If $\alpha_G < 1$, we obtain from eqs. (2.22), (3.7) that $\alpha_F^{(2)} = \alpha_G$. Then,

$$-\frac{(A(\mu^2))^3 B(\mu^2)}{A_2(\mu^2)} N_C g_R^2(\mu^2) \widetilde{Z}_1 \ \phi(0, \alpha_G) \simeq 16\pi^2, \tag{3.8}$$

where, according to eqs. (2.19), (2.21) $\phi(0, \alpha_G)$ is given by

$$\phi(0, \alpha_G) = \frac{3}{2\alpha_G(\alpha_G + 1)(\alpha_G + 2)(1 - \alpha_G)} \tag{3.9}$$

- Similarly if $\alpha_G = 1$, eq. (2.24) applied to eq. (3.7) leads to

$$\frac{(A(\mu^2))^3 B(\mu^2)}{A_2(\mu^2)} N_C g_R^2(\mu^2) \widetilde{Z}_1 \ \simeq 64\pi^2. \tag{3.10}$$

- At last, if $\alpha_G > 1$, eqs. (2.25) and (3.7) imply: $\alpha_F^{(2)} = 1$. i.e., a ghost dressing function which behaves quadratically for small momenta, In this case, however, as already said the ghost loop cannot be evaluated using the IR power laws over the whole integration range and it is therefore not possible to solve the GPDSE consistently, nor even to determine the small-momentum behaviour of the dressing functions, without matching appropriately those power laws to the UV perturbative formulas. Thus, we are not able to derive a constraint for the next-to-leading coefficient, $A_2(\mu^2)$.

In summary, the GPDSE admits IR solutions with $\alpha_F = 0$ and any $\alpha_G > 0$, provided that

$$F_{\mathrm{IR}}(q^2, \mu^2) = \begin{cases} A(\mu^2) \left(1 - \phi(0, \alpha_G) \dfrac{\widetilde{g}^2(\mu^2)}{16\pi^2} A^2(\mu^2) B(\mu^2)(q^2)^{\alpha_G} \right) & \alpha_G < 1 \\[2mm] A(\mu^2) \left(1 + \dfrac{\widetilde{g}^2(\mu^2)}{64\pi^2} A^2(\mu^2) B(\mu^2) q^2 \ln q^2 \right) & \alpha_G = 1 \\[2mm] A(\mu^2) + A_2(\mu^2) q^2 & \alpha_G > 1 \end{cases} \tag{3.11}$$

where $\widetilde{g}^2(\mu^2) = N_C g_R^2(\mu^2)\widetilde{Z_1}$ and $\phi(0, \alpha_G)$ is given in eq. (3.9). The gluon dressing function is supposed to behave as indicated in eq. (2.12). In particular for $\alpha_G = 1$, the gluon propagator takes a finite (and non-zero) value at zero momentum, $B(\mu^2)$, after applying MOM renormalisation prescription at $q^2 = \mu^2$.

4. The ghost-gluon and three-gluon Ward-Slavnov-Taylor identity

In the previous section, we have analysed the infrared behaviour of GPDSE solutions and found that the ghost dressing function can either diverge at vanishing momentum ($\alpha_F = -\alpha_G/2$ with $\alpha_G > 0$) or give a finite value ($\alpha_F = 0$ with any $\alpha_G > 0$). As appendix A shows, the GPDSE can be derived from the general Ward-Slavnov-Taylor equation [14]. The Ward-Slavnov-Taylor identities (WSTI) can be derived formally from the gauge invariance of the path integral, eq. (A.1) of A, as shown in [15]. This is the case in lattice simulations. If the path integral is limited to a domain of the configuration space such as in the Gribov-Zwanziger approach, used in [16], the STI may be broken. We assume a gauge invariant path integral and will now invoke the WSTI for general covariant gauges relating the 3-gluon, $\Gamma_{\lambda\mu\nu}(p, q, r)$, and ghost-gluon vertices,

$$
\begin{aligned}
p_\lambda \Gamma_{\lambda\mu\nu}(p, q, r) = {} & \frac{F(p^2)}{G(r^2)}(\delta_{\rho\nu}r^2 - r_\rho r_\nu)\widetilde{\Gamma}_{\rho\mu}(r, p; q) \\
& - \frac{F(p^2)}{G(q^2)}(\delta_{\rho\mu}q^2 - q_\rho q_\mu)\widetilde{\Gamma}_{\rho\nu}(q, p; r).
\end{aligned}
\tag{4.1}
$$

to shed some light on that matter [10, 17]. Using for the ghost-gluon vertex the general decomposition[5] [18]

$$
\begin{aligned}
\widetilde{\Gamma}_{\nu\mu}(p, q; r) = {} & \delta_{\nu\mu}a(p, q; r) - r_\nu q_\mu b(p, q; r) + p_\nu r_\mu c(p, q; r) \\
& + r_\nu p_\mu d(p, q; r) + p_\nu p_\mu e(p, q; r),
\end{aligned}
\tag{4.2}
$$

and multiplying by r_ν both sides of eq. (4.1), one obtains:

$$
r_\nu p_\lambda \Gamma_{\lambda\mu\nu}(p, q, r) = \frac{F(p^2)}{G(q^2)} X(q, p; r)\left[(q \cdot r)q_\mu - q^2 r_\mu\right];
\tag{4.3}
$$

where

$$
X(q, p; r) = a(q, p; r) - (r \cdot p)b(q, p; r) + (r \cdot q)d(q, p; r).
\tag{4.4}
$$

Since the vertex function, Γ, in the l.h.s. of eq. (4.3) is antisymmetric under $p \leftrightarrow r$ and $\lambda \leftrightarrow \nu$, one can then conclude that [10, 19]:

$$
F(p^2)X(q, p; r) = F(r^2)X(q, r; p).
\tag{4.5}
$$

This last result is a compatibility condition required for the WSTI to be satisfied that does not involve the 3-gluon vertex and implies a strong correlation between the infrared

---

[5]We work of course on the energy-momentum shell, so that the relation $p + q + r \equiv 0$ holds

behaviours of the ghost-gluon vertex and the ghost propagator. Now, under the only additional hypothesis that those scalars of the ghost-gluon vertex decomposition in eq. (4.2) contributing to the scalar function X defined in eq. (4.4) are regular[6] when one of their arguments goes to zero while the others are kept non-vanishing, one can consider the small p limit in eq. (4.5) and obtain:

$$F(p^2)X(q,0;-q) = F(q^2)X(q,-q;0) + \mathcal{O}(p^2) \tag{4.6}$$

This has to be true for **any** value of q, which implies $F(p^2)$ **goes to some finite and non-zero value when p goes to zero**, since neither $X(q,0;-q)$ nor $X(q,-q;0)$ are presumably zero for all values of q. Rephrased in terms of infrared exponents, the latter argument implies that $\alpha_F = 0$.

To reach the above conclusions we did not appeal to the properties of the 3-gluon vertex, apart from the symmetry under the exchange of gluon legs. If one assumes in addition that the longitudinal part of the 3-gluon vertex also behaves regularly when anyone of its arguments goes to 0, the others being kept non-vanishing, a divergent gluon propagator at vanishing momentum will be implied [3, 8, 10]. Of course, as far as it involves a vertex with longitudinal gluons which have not been very extensively studied, this last conclusion is not as clean as the previous one about the ghost dressing (according to authors of ref. [4] a soft kinematical singularity appears for the landau-gauge 3-gluon vertex, however it does not concern our proof relying on the regularity of the longitudinal-longitudinal-transverse 3-gluon vertex).

In ref. [10], we showed that only a very mild divergence, for example of logarithmic type, could be compatible with current LQCD results for the gluon propagator. The IR analysis of the previous section can be straightforwardly extended to this case by generalizing

$$G_{\text{IR}}(q^2,\mu^2) = B(\mu^2)\left(q^2\right)^{\alpha_G} \log^\nu\left(\frac{1}{q^2}\right), \tag{4.7}$$

the effect of which is to modify eq. (3.11) with

$$F_{\text{IR}}(q^2,\mu^2) = \begin{cases} A(\mu^2)\left(1-\phi(0,\alpha_G)\dfrac{\widetilde{g}^2(\mu^2)}{16\pi^2}A^2(\mu^2)B(\mu^2)(q^2)^{\alpha_G}\log^\nu\left(q^{-2}\right)\right) & \alpha_G < 1 \\[2ex] A(\mu^2)\left(1-\dfrac{\widetilde{g}^2(\mu^2)}{(\nu+1)64\pi^2}A^2(\mu^2)B(\mu^2)q^2\log^{(\nu+1)}\left(q^{-2}\right)\right) & \alpha_G = 1 \\[2ex] A(\mu^2) + A_2(\mu^2)q^2\log^\nu\left(q^{-2}\right) & \alpha_G > 1 \end{cases} \tag{4.8}$$

where only the power of the logarithm is then modified.

Sticking now to the case where α_F is zero (for the reasons explained above) and α_G is 1 (as suggested by the lattice results) we are left with

$$F_{\text{IR}}(q^2,\mu^2) = F_{\text{IR}}(0,\mu^2)\left(1-\frac{\widetilde{g}^2(\mu^2)}{(\nu+1)64\pi^2}F_{\text{IR}}(0,\mu^2)^2 B(\mu^2)q^2\log^{(\nu+1)}\left(\frac{M^2}{q^2}\right)\right), \tag{4.9}$$

---

[6]Note also that, for our purposes, it will actually be enough to restrict, and not forbid, the possible presence of singularities in the scalar coefficient functions provided that they could be compensated by kinematical zeroes stemming from the tensors.

according to whether there are logarithmic corrections to the gluon propagator ($\nu \neq 0$) or not ($\nu = 0$). Here, M is some scale which is out of the scope of the IR analysis we performed in the previous section and, if $\nu = 0$, $B(\mu^2) = G^{(2)}_{\rm IR}(0, \mu^2)$ is the gluon propagator at zero momentum.

5. Ghost propagator from LQCD

The theoretical study by Zwanziger [20] of the Faddeev-Popov operator on the lattice in Landau gauge triggered the first Lattice simulation of the ghost propagator [21] in SU(2) and SU(3) gauge theories and the subsequent activity which, mainly for technical reasons, was mostly dedicated to the SU(2) lattice gauge theory in the infrared region. It was only in the last few years that several studies of the SU(3) ghost propagator focused on its infrared region and the Gribov copy problem [2] or on their perturbative [22, 23] and OPE non-perturbative [24] descriptions. An unambiguous consensus from LQCD, after all this work, pointed that $F^2(q^2)G(q^2) \to 0$ when $q \to 0$ (see, for instance, [2, 3]) and, consequently, that the solution I is excluded provided that the finite-volume artefacts are indeed under control. As a matter of the fact, finite-volume lattice simulations all agree on a ghost propagator pretty close to the tree-level ($\alpha_F \simeq 0$) and a gluon propagator not far from being a constant at vanishing momentum ($\alpha_G \simeq 1$).

Very recently [5, 6], simulations on large volumes lattices (with a fair control over the finite-volume lattice artifacts) yielded solutions for the ghost dressing function confirming that α_F is indeed in the vicinity of 0. Let us now briefly comment about the ghost propagator results from these two papers:

- The authors of ref. [5] simulated the ghost propagator in $56^4, 64^4, 72^4, 80^4$ volumes with an impressive control of the finite-size effects over a huge momentum range from $q^2 \simeq 0.01\,{\rm GeV}^2$ to $q^2 \simeq 10\,{\rm GeV}^2$. They fit an IR exponent, $\alpha_F = -0.174$, that appears to be in the vicinity of zero (but negative) and at least much larger that the most frequently advocated value ($\simeq -0.5$). The fit is however delicate because the power behavior is dominant, if ever, only on a very small momentum domain.[7] Indeed, it is adviseable to try a fitting function inspired from eq. (4.8). Moreover, the numerical solution (type II), obtained in ref. [9], after a rescaling because of the MOM renormalisation, describes strikingly well the lattice ghost propagator data from ref. [5] over a large momentum window, from $0.05\,{\rm GeV}$ to $3\,{\rm GeV}$.

- The authors of ref. [6] computed an IR ghost propagator exponent, $a_G(= -\alpha_F)$, for several 3-dimensional and 4-dimensional lattice volumes (ranging from 140^3 to 320^3 and from 48^4 to 128^4) and collected the results in their table 1. The values of α_F from that table are not only in the vicinity of zero (although being negative) but they approach systematically zero when the volume increases. They fit the power behaviour on a small domain with two or four momentum data.

---

[7]The fitted IR exponent is unstable, lying more and more in the vicinity of zero as the momentum domain becomes smaller (see figure 2 of ref. [5]).

In ref. [9], we showed that the $k^2 \log(k^2)$ term given by eq. (4.9) describes very well the behaviour of a numerical solution of the GPDSE, eq. (2.4), for $\widetilde{g}^2(\mu = 1.5 \text{ GeV}) = 29$ (such a value corresponds to the best description of our ghost propagator lattice data) and with a gluon dressing function taken from a lattice simulation (see figure 1 of ref. [9]). We showed at the same time that including a logarithmic divergence changes appreciably neither the deduced ghost propagator nor the conclusions about the infrared solutions.

In this same work we analysed in detail the behaviour of the numerical solutions of the GPDSE as functions of $\widetilde{g}^2(\mu = 1.5 \text{ GeV})$ and discovered that a singular solution, behaving as $1/q$ for small momentum (as the relation $2\alpha_F + \alpha_G = 0$ requires), appeared only for the specific value $\widetilde{g}^2(\mu = 1.5 \text{ GeV}) = 33.198\ldots$. This solution belongs evidently to what is referred to above as class I, with $\alpha_F = -1/2$ and does satisfy the relation $2\alpha_F + \alpha_G = 0$. Furthermore, the closer $\widetilde{g}^2(\mu^2)$ to this critical value, the smaller the region near $q = 0$ where eq. (4.9) is valid.

6. Discussion and conclusions

Thus the present analytical considerations and the previous numerical study converge towards a consistent description of the set of solutions of the ghost Dyson-Schwinger equations:

- A class of solutions where the ghost dressing function is finite and non zero at $q^2 = 0$ (i.e. $\alpha_F = 0$), depending continuously on the coupling constant (or equivalently on $F(0)$). Those solutions do not fulfill the relation $2\alpha_F + \alpha_G = 0$ but appear, for an appropriate value of the coupling, to be in very good agreement with the lattice results.

- An exceptional solution, obtained for a critical value of the coupling is IR-divergent with $\alpha_F = -1/2$. Contrary to the previous ones it satisfies $2\alpha_F + \alpha_G = 0$ but is in clear disagreement with the lattice data over a large range of momenta.

We have demonstrated that the discrepancy between LQCD results (implying unambiguously that $\alpha_F \simeq 0$ and $\alpha_G \simeq 1$) and the usual DSE solutions ($2\alpha_F + \alpha_G = 0$) can be solved if the second type (II) of solutions is considered. The existence of this second class besides the usual solution (type I) has been proven after carefully renormalising the GPDSE and applying a subtraction procedure to deal with the remaining (after renormalisation) UV singularity. This new solution yields a finite ghost dressing function at vanishing momentum while $F^2(q^2)G(q^2)$ goes to a zero when $q \to 0$ contrary to what occurs with type I.

For this (type II) solution, an asymptotic formula of the ghost dressing function is obtained that only depends on the IR one for the gluon which is taken as an *ansatz* in this exercise and on the renormalized coupling. The numerical analysis of the GPDSE in ref. [9] proves that the type II solution exists for any coupling below a given critical value and that it verifies the asymptotic formula.

The WSTI involving the 3-gluon and the ghost-gluon vertex is particularly useful to gain some knowledge about the ghost dressing function: by simply assuming the regularity

of some of the tensorial components of the ghost-gluon vertex, one can conclude that the ghost dressing function is finite and non zero at vanishing momentun. Then, WSTI with the mentioned regularity assumption will discard the solution of type I.

Furthermore, LQCD data point to $2\alpha_F + \alpha_G \simeq 1$ (certainly larger than 0). Would one wish to reconcile these data with type I solution ($2\alpha_F + \alpha_G = 0$), very strong finite-volume artifacts would be needed. Such a finite-size effect should strengthen the divergence of a ghost propagator behaving at finite-volume like at tree-level and damp to zero the gluon propagator. This is very doubtful considering that sizeable discrepancies between lattice and solution I appear at momenta of the order of $\simeq 0.3$ GeV. On the contrary, very recent LQCD data in large volumes [5, 6] show a fair stability as the volume increases and, if any, a trend towards solution II ($\alpha_F = 0$). This is confirmed by the numerical analysis of ref. [9] which proves that both type I and II solutions *live* at infinite volume for different values of the coupling constant.

It is worth also pointing that some attempts to accomodate lattice data within DS coupled equations [25] and within the Gribov-Zwanziger approach [16] led to solutions for gluon and ghost propagators that behave pretty much like our solution II does.

Altogether we strongly believe that the question of the ghost propagator behaviour at small momentum is essentially solved. The solution type II of GPDSE avoiding the previous discrepancies, the three methods (DSE, WSTI and LQCD) strikingly converge to the same result: **a finite ghost dressing function at vanishing momentum**. The case of the gluon propagator needs further study.

Acknowledgments

A. The Dyson-Schwinger equation as a Ward-Slavnov-Taylor identity

A very general method to derive Slavnov-Taylor identities consists in taking advantage of the transformation properties of

$$e^{G(J)} = \int \mathcal{D}(A) \det \mathcal{M} \exp\left[i \int d^4x \left(\mathcal{L} - \frac{1}{2\alpha}(\partial_\mu A_\mu^a)(\partial_\mu A_\mu^a) + J_\mu^a A_a^\mu\right)\right] \qquad \text{(A.1)}$$

under gauge transformations (cf. [15]).

$\mathcal{M}$ is the Faddeev-Popov operator and the notation $<, >_J$ indicates that the source term J has to be kept, although it will eventually be set to 0 (this is denoted in the following by the suppression of the J subscript). Taking the derivative of the gauge transformed of eq. (A.1) with respect to the gauge parameters leads to the general Slavov-Taylor equation

$$\frac{1}{\alpha} < (\partial_\mu A_\mu^a(x)) >_J = < \int d^4y J_\mu^c(y) \, D_\mu^{cb}(y) F^{(2)ba}(y,x) >_J . \qquad \text{(A.2)}$$

$F^{(2)ba}(y,x)$ is the ghost propagator and its presence here is simply due to its very definition as the inverse of the Faddeev-Popov operator. If one derives eq. (A.2) with respect to $J_\rho^d(z)$ one gets:

$$\frac{1}{\alpha} < (\partial_\mu A_\mu^a(x)) A_\rho^d(z) >_J \; = \; < D_\rho^{db}(z) F^{(2)ba}(z,x) >_J$$
$$+ < \int d^4 y J_\mu^c(y) \, D_\mu^{cb}(y) F^{(2)ba}(y,x) A_\rho^d(z) >_J . \quad (A.3)$$

A first consequence of this relation is the triviality of the longitudinal gluon propagator. To see this, it suffices to derive both its sides with respect to z_ρ and to set J to zero. The result is

$$\frac{1}{\alpha} < (\partial_\mu A_\mu^a(x))(\partial_\rho A_\rho^d(z)) > \; = \; < \partial_\rho D_\rho^{db}(z) \, F^{(2)ba}(z,x) >$$
$$= \delta_{ad} \, \delta_4(z-x). \quad (A.4)$$

In order to derive the second line we have invoked the fact that $\partial_\rho D_\rho^{db}(z)$, the Faddeev-Popov operator, is the inverse of the ghost propagator $F^{(2)}$. Thus, in momentum space, the general form of the gluon propagator for an arbitrary covariant gauge reads

$$G_{\mu\nu}^{(2)ab}(q) = \delta^{ab} \left[G^{(2)}(q^2) \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) + \alpha \frac{q_\mu q_\nu}{(q^2)^2} \right]. \quad (A.5)$$

Turning now back to eq. (A.3) and letting J go to zero we obtain

$$\frac{1}{\alpha} < (\partial_\mu A_\mu^a(x)) A_\rho^d(z) > = < D_\rho^{db}(z) F^{(2)ba}(z,x) > \quad (A.6)$$

which is nothing else than the GPDSE. Actually its l.h.s. involves only the longitudinal part of the gluon propagator, that we have just seen to be trivial:

$$\frac{1}{\alpha} < (\partial_\mu A_\mu^a(x)) A_\rho^d(z) > = \partial_\rho \Box^{-1}(x,z); \quad (A.7)$$

as for the r.h.s it can be rewritten as:

$$< D_\rho^{db}(z) F^{(2)ba}(z,x) > = < \partial_\rho F^{(2)da}(z,x) > + i < gf^{deb} A_\rho^e(z) F^{(2)ba}(z,x) > . \quad (A.8)$$

The 3-point gluon-ghost Green's function can be expressed in terms of vertex functions and propagators through

$$\widetilde{G}_\rho^{(3)fgh}(p,q,r) \equiv -i \int d^4 x d^4 t d^4 z e^{ipx} e^{irz} e^{iqt} < A_\rho^f(t) F^{(2)gh}(z,x) > \quad (A.9)$$

$$= g \frac{F(p^2)}{p^2} \frac{F(r^2)}{r^2} \left[\frac{G(q^2)}{q^2} \left(\delta_{\rho\nu} - \frac{q_\rho q_\nu}{q^2} \right) + \alpha \frac{q_\rho q_\nu}{(q^2)^2} \right] \times \quad (A.10)$$

$$\times f^{fgh} \widetilde{\Gamma}_\nu(p,r;q)(2\pi)^4 \delta_4(p+q+r)$$

Now, we Fourier-transform eq. (A.6), use eqs. (A.7)–(A.9) and obtain

$$\frac{k_\rho}{k^2} = \frac{k_\rho}{k^2} F(k^2) - gf^{deb} f^{eba} \int \frac{d^4 q}{(2\pi)^4} \frac{F(k^2)}{k^2} \frac{F((k+q)^2)}{(k+q)^2} \times$$
$$\times \left[\frac{G(q^2)}{q^2} \left(\delta_{\rho\nu} - \frac{q_\rho q_\nu}{q^2} \right) + \alpha \frac{q_\rho q_\nu}{(q^2)^2} \right] \widetilde{\Gamma}_\nu(k,-k-q;q), \quad (A.11)$$

where the usual form of GPDSE can be recovered from by multiplying with k_ρ and dividing by $F(k^2)$, which leads to

$$F^{-1}(k^2) = 1 - g f^{deb} f^{eba} \int \frac{d^4q}{(2\pi)^4} \frac{F((k+q)^2)}{(k+q)^2} \times$$
$$\times \left[\frac{G(q^2)}{q^2} \left(k_\nu - \frac{(qk)q_\nu}{q^2} \right) + \alpha \frac{(qk)q_\nu}{(q^2)^2} \right] \widetilde{\Gamma}_\nu(k, -k-q; q). \qquad \text{(A.12)}$$

This is a general result, valid in any covariant gauge. Of course the α-depending (longitudinal) term disappears in Landau gauge. $\widetilde{\Gamma}_\nu(k, -k-q; q)$ is related to the quantity, $\widetilde{\Gamma}_{\mu\nu}$, that was previously introduced in sections 2 and 4 through

$$\widetilde{\Gamma}_\nu(k, -k-q; q) = -ig k_\mu \widetilde{\Gamma}_{\mu\nu}(k, -k-q; q)$$

and it is usually decomposed into $\widetilde{\Gamma}_\nu(k, -k-q; q) = g\left[k_\nu H_1(k,q) + q_\nu H_2(k,q)\right]$. After inserting this in eq. (A.12) and restricting to the Landau gauge case one finally obtains

$$F^{-1}(k^2) = 1 + g^2 N_c \int \frac{d^4q}{(2\pi)^4} \frac{F((k+q)^2)}{(k+q)^2} \left[\frac{G(q^2)}{q^2} \left(\frac{(qk)^2}{q^2} - k^2 \right) \right] H_1(k,q). \qquad \text{(A.13)}$$

References

[1] R. Alkofer and L. von Smekal, *The infrared behavior of QCD Green's functions: confinement, dynamical symmetry breaking and hadrons as relativistic bound states*, Phys. Rept. **353** (2001) 281 [hep-ph/0007355].

[2] A. Sternbeck, E.M. Ilgenfritz, M. Muller-Preussker and A. Schiller, *The gluon and ghost propagator and the influence of Gribov copies*, Nucl. Phys. **140** (Proc. Suppl.) (2005) 653 [hep-lat/0409125]; *The influence of Gribov copies on the gluon and ghost propagator*, AIP Conf. Proc. **756** (2005) 284 [hep-lat/0412011].

[3] P. Boucaud et al., *The infrared behaviour of the pure Yang-Mills Green functions*, hep-ph/0507104.

[4] R. Alkofer, M.Q. Huber and K. Schwenzer, *On infrared singularities in Landau gauge Yang-Mills theory*, arXiv:0801.2762.

[5] I.L. Bogolubsky, E.M. Ilgenfritz, M. Muller-Preussker and A. Sternbeck, *The Landau gauge gluon and ghost propagators in 4D* SU(3) *gluodynamics in large lattice volumes*, PoS(LATTICE 2007)290 [arXiv:0710.1968].

[6] A. Cucchieri and T. Mendes, *What's up with IR gluon and ghost propagators in Landau gauge? A puzzling answer from huge lattices*, PoS(LATTICE 2007)297 [arXiv:0710.0412].

[7] P. Boucaud et al., *Is the QCD ghost dressing function finite at zero momentum?*, JHEP **06** (2006) 001 [hep-ph/0604056].

[8] P. Boucaud et al., *Constraints on the IR behaviour of gluon and ghost propagator from Ward-Slavnov-Taylor identities*, Eur. Phys. J. **A31** (2007) 750 [hep-ph/0701114].

[9] P. Boucaud et al., *IR finiteness of the ghost dressing function from numerical resolution of the ghost SD equation*, arXiv:0801.2721.

[10] P. Boucaud et al., *Divergent IR gluon propagator from Ward-Slavnov-Taylor identities?*, *JHEP* **03** (2007) 076 [`hep-ph/0702092`].

[11] F.D.R. Bonnet, P.O. Bowman, D.B. Leinweber, A.G. Williams and J.M. Zanotti, *Infinite volume and continuum limits of the Landau-gauge gluon propagator*, *Phys. Rev.* **D 64** (2001) 034501 [`hep-lat/0101013`].

[12] C. Lerche and L. von Smekal, *On the infrared exponent for gluon and ghost propagation in Landau gauge QCD*, *Phys. Rev.* **D 65** (2002) 125006 [`hep-ph/0202194`].

[13] J.C.R. Bloch, *Two-loop improved truncation of the ghost-gluon Dyson- Schwinger equations: multiplicatively renormalizable propagators and nonperturbative running coupling*, *Few Body Syst.* **33** (2003) 111 [`hep-ph/0303125`].

[14] J.C. Taylor, *Ward identities and charge renormalization of the Yang-Mills field*, *Nucl. Phys.* **B 33** (1971) 436;
A.A. Slavnov, *Ward identities in gauge theories*, *Theor. Math. Phys.* **10** (1972) 99 [*Teor. Mat. Fiz.* **10** (1972) 153].

[15] C. Itzykson and J.-B. Zuber, *Quantum field theory*, McGraw-Hill ed. U.S.A (1980).

[16] D. Dudal, S.P. Sorella, N. Vandersickel and H. Verschelde, *New features of the gluon and ghost propagator in the infrared region from the Gribov-Zwanziger approach*, `arXiv:0711.4496`.

[17] A.I. Davydychev, P. Osland and O.V. Tarasov, *Three-gluon vertex in arbitrary gauge and dimension*, *Phys. Rev.* **D 54** (1996) 4087 [*Erratum ibid.* **59** (1999) 109901] [`hep-ph/9605348`].

[18] J.S. Ball and T.W. Chiu, *Analytic properties of the vertex function in gauge theories. 2*, *Phys. Rev.* **D 22** (1980) 2550 [*Erratum ibid.* **23** (1981) 3805].

[19] K.G. Chetyrkin and A. Retey, *Three-loop three-linear vertices and four-loop MOM β-functions in massless QCD*, `hep-ph/0007088`.

[20] D. Zwanziger, *Fundamental modular region, Boltzmann factor and area law in lattice gauge theory*, *Nucl. Phys.* **B 412** (1994) 657.

[21] H. Suman and K. Schilling, *First lattice study of ghost propagators in* SU(2) *and* SU(3) *gauge theories*, *Phys. Lett.* **B 373** (1996) 314 [`hep-lat/9512003`].

[22] S. Furui and H. Nakajima, *Infrared features of the Landau gauge QCD*, *Phys. Rev.* **D 69** (2004) 074505 [`hep-lat/0305010`]; *What the Gribov copy tells on the confinement and the theory of dynamical chiral symmetry breaking*, *Phys. Rev.* **D 70** (2004) 094504 [`hep-lat/0403021`].

[23] P. Boucaud et al., *Asymptotic behavior of the ghost propagator in* SU(3) *lattice gauge theory*, *Phys. Rev.* **D 72** (2005) 114503 [`hep-lat/0506031`].

[24] P. Boucaud et al., *Non-perturbative power corrections to ghost and gluon propagators*, *JHEP* **01** (2006) 037 [`hep-lat/0507005`].

[25] A.C. Aguilar and A.A. Natale, *A dynamical gluon mass solution in a coupled system of the Schwinger-Dyson equations*, *JHEP* **08** (2004) 057 [`hep-ph/0408254`];
A.C. Aguilar and J. Papavassiliou, *Gluon mass generation in the PT-BFM scheme*, *JHEP* **12** (2006) 012 [`hep-ph/0610040`]; *On dynamical gluon mass generation*, *Eur. Phys. J.* **A31** (2007) 742 [`arXiv:0704.2308`].